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The flow of a viscous liquid down a variable incline

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SUMMARY
The flow of a thin film of almost inviscid fluid down a slope of variable inclination is considered. The equations governing the flow are the boundary-layer equations, and a numerical solution of these equations is obtained. It is found that in cases where the liquid film attempts to flow against gravity the flow will separate. The numerical solution indicates a singularity at the separation point. The nature of this singularity is discussed.

1. Introduction

The problem considered in this paper is that of the flow of a thin film of almost inviscid liquid down a variable incline. The thickness of the film is assumed small compared to a, a typical radius of curvature of the bed of the incline, while the Reynolds number, based on a, is assumed large. The equations governing the flow will then be the boundary-layer equations, with the difference that the flow in this case being driven by the gravitational force and not the usual pressure gradient. The outer boundary conditions are applied on a variable free surface. The form that this free surface takes is one of the unknowns in the problem.

The case of a horizontal bed has been discussed by Watson [1]. He obtained a similarity solution for the problem, and discussed how this is set up from certain initial conditions. Eventually there will be a hydraulic jump and Watson gave an estimate of the position of this jump.

Ackerberg [2] has treated the flow of a thin film down a vertical slope. He took for the initial velocity one with a uniform profile and showed that the film rapidly settled down to one of constant thickness, and, in a sense, "forgot" the way in which the flow was set up. Ackerberg's analysis was done for a vertical slope, but can be taken over directly to a slope making a constant angle with the vertical.

Smith [3] extended Ackerberg's work by assuming that, after the flow had reached its fully developed state, the inclination of the slope varied in a prescribed manner. He found that, as the bed became more horizontal, the gravity forces were less effective in driving the flow and the liquid in the film was slowed down more by viscosity. In certain cases this could lead to the flow reaching a separation point (i.e. a point where the skin friction became zero). Smith [3] used a momentum integral method to solve the problem. He chose fourth order polynomials for the velocity profile and obtained two ordinary differential equations (one from an integrated form of the boundary-layer equations and the other from a continuity condition). On solving these equations numerically he found that the solution broke down in the cases where separation was indicated. He then conjectured that this breakdown was due to the singular nature of the solution near the separation point. The main purpose of this paper is to confirm this conjecture. To do this, a numerical solution of the full equations is obtained. It is found that, in the cases where the flow separates there is a singularity in the numerical solution at the separation point. The nature of this singularity is discussed, and it is shown that the skin friction τ_ω behaves like $(x_s-x)^{\frac{1}{2}}$ and the film thickness H like $H_s+(x_s-x)^{\frac{1}{2}}H_1$ near the separation point (x is the co-ordinate that measures distance along the bed, $x=x_s$ is the separation point, and H_s and H_1 are constant). This behaviour explains why Smith's approximate solution broke down. Derivatives of H occurred in his equations.

The bed profiles considered here are those used by Smith [3], so that a comparison can be made with his work. The numerical solution shows that separation will occur only where gravity is acting against the flow, whereas Smith's method gave separation in cases where the effect of gravity, though reduced, was still acting in the direction of the flow.

2. Formulation

We are going to consider the steady flow under gravity of a thin film of almost inviscid liquid down a slope of variable inclination. We define a co-ordinate system such that x measures distance along the bed and y distance normal to it. Call the angle that the bed makes with the horizontal $\theta(x)$ and the free surface profile $y=h(x)$. Two types of slope are to be considered. The first consists of a straight section inclined at the constant angle θ_0 to the horizontal for $x<0$ and a circular arc of radius a for $x\geqq 0$ with a smooth changeover at $x=0$ so that the tangent to the arc makes the angle θ_0 with the horizontal at $x=0$. In this case $\sin\theta=\sin(\theta_0-x/a)$ for $x\geqq 0$. The second profile is that where $\theta(x)$ is given by $\sin\theta=S_n(x)=\frac{1}{2}(1-nx/(1+x^2))$, n being a positive integer. Both these forms for the bed were treated by Smith [3].

If h_0, a and U_0 are typical values of the film thickness, radius of curvature of the bed and streamwise velocity respectively then we are considering the situation where $\beta=h_0/a\ll 1$, the Reynolds number $R_e=U_0a/\nu\gg 1$ while $\varepsilon=U_0^2/ga$ is $O(1)$. (g is the acceleration of gravity and ν is the kinematic viscosity of the liquid). If we define a Froude number by $F_r=U_0^2/gh_0$ then this last assumption can be written as $\varepsilon=\beta F_r=O(1)$. Under these assumptions the equations governing the flow will be, $t_0=O(\beta)$, the boundary-layer equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=gS(x)+\nu\frac{\partial^2 u}{\partial y^2}\,. \tag{2}$$

u and v are the velocity components in the x- and y-directions respectively and $S(x)=\sin\theta$. The boundary conditions are

$$u=v=0 \text{ on } y=0\,,\quad \frac{\partial u}{\partial y}=0 \text{ on } y=h(x)\,,$$
$$\int_0^{h(x)} u\,dy=Q \tag{3}$$

(where Q is a constant and is the total volume flux).

To specify the problem completely, initial conditions are also required. For both types of slope considered the flow starts down a slope of constant inclination θ_0, in the first case it is for $x<0$ and in the second $\theta_0=\pi/6$ as $x\to-\infty$. For a constant slope, equations (1) and (2) possess the well-known solution

$$u=g\,\frac{\sin\theta_0}{\nu}\left(hy-\frac{y^2}{2}\right),\qquad v=0\,. \tag{4}$$

The flow is specified by the total flow Q, so that, in this case, $h=(3\nu Q/g\sin\theta_0)^{\frac{1}{3}}$. From this solution we get h_0 and U_0, the typical values of the film thickness and streamwise velocity, as

$$h_0=(\nu Q/g)^{\frac{1}{3}}\quad\text{and}\quad U_0=gh_0^2/\nu\,, \tag{5}$$

so that $\varepsilon=(Q^4/a^3\nu^2g)^{\frac{1}{3}}$. The required initial conditions are then given by (4), and are taken at $x=0$ in the first case and as $x\to-\infty$ in the second. Ackerberg [2] has shown that for a film flowing down a plate (4) is strictly an asymptotic solution, but it is attained very quickly. In fact the error involved is exponentially small. This is the justification for using (4) as the initial conditions.

From equation (1) we can define a stream function ψ in the usual way, and then introduce non-dimensional variables by writing $X=x/a$, $Y=y/h_0$, $\psi=U_0 h_0 \Psi$ and $H=h/h_0$. Equation (2) becomes, on using (5)

$$\frac{\partial^3 \Psi}{\partial Y^3} + S(X) = \varepsilon \left(\frac{\partial \Psi}{\partial Y} \frac{\partial^2 \Psi}{\partial X \partial Y} - \frac{\partial \Psi}{\partial X} \frac{\partial^2 \Psi}{\partial Y^2} \right) \tag{6}$$

with

$$\Psi = \frac{\partial \Psi}{\partial Y} = 0 \text{ on } Y=0, \quad \Psi=1, \quad \frac{\partial^2 \Psi}{\partial Y^2} = 0 \text{ on } Y=H(X), \tag{7}$$

together with initial conditions

$$\Psi = \sin \theta_0 \, Y^2 (3H_i - Y)/6, \tag{8}$$

where $H_i = (3/\sin \theta_0)^{\frac{1}{3}}$.

3. Numerical solution

In this section we give a method of obtaining a numerical solution of equation (6) together with boundary conditions (7) which starts with the given initial conditions (8) and proceeds step-by-step downstream until the separation point is reached. One of the problems in solving (6) is that the outer boundary conditions are applied on $Y=H(X)$ and $H(X)$ is a variable in the problem. One method of overcoming this difficulty, as used by Ackerberg [2], is to use the von Mises form of the boundary-layer equations i.e. use (X, Ψ) as independent variables instead of (X, Y). This has the advantage that the boundary conditions are now applied on $\Psi=0$ and $\Psi=1$ and H does not appear explicitly in the equations. This method has the drawback, however, in that the solution is singular near $\Psi=0$, in fact $u \propto \Psi^{\frac{1}{2}}$ for small Ψ. This can be overcome but not without some difficulty, but even then the variations in velocity are confined to a very small region near $\Psi=0$. This suggests trying to set up a numerical procedure for solving (6) using a more "natural" co-ordinate system.

First make the transformation $\Psi=Hf(\eta, X)$, $\eta=Y/H(X)$, and writing $q \equiv \partial f/\partial \eta$, (6) and (7) become

$$\frac{\partial^2 q}{\partial \eta^2} + H^2 S + \varepsilon \frac{\partial q}{\partial \eta} \int_0^\eta \left(H \frac{dH}{dX} q + H^2 \frac{\partial q}{dX} \right) d\eta - \varepsilon H^2 q \frac{\partial q}{\partial X} = 0 \tag{9}$$

$$H \int_0^1 q \, d\eta = 1 \tag{10}$$

with boundary conditions

$$q=0 \text{ on } \eta=0, \quad \frac{\partial q}{\partial \eta} = 0 \text{ on } \eta=1 \tag{11}$$

and initial conditions $q=H_i \sin \theta_0 (\eta - \eta^2/2)$.

The boundary conditions are now applied on fixed lines but the variable $H(X)$ appears in the equations, and the equation expressing constancy of volume flow (equation (10)) is now coupled with the momentum equation, equation (9).

To solve equations (9) and (10) numerically, we start with the initial values of q and H and proceed step-by-step downstream. The idea is that, assuming q and H known at X_1, to give a method for calculating them at X_2, with $X_2 > X_1$. To do this, derivatives in the X-direction are replaced by differences and all other terms are averaged over the step length $\Delta X = X_2 - X_1$. Equations (9) and (10) then become a system of non-linear ordinary differential equations for the unknowns q_2 and H_2 (the suffices 1 and 2 denote values of the functions at X_1 and X_2 respectively). It is, however, more convenient to work in terms of $\omega = q_1 + q_2$. To solve these ordinary differential equations the range $0 \leqq \eta \leqq 1$ is divided up into N equal divisions each of length k and derivatives in the η-direction are replaced by finite differences of step length k.

From (9) and (10) we then get the $N+1$ non-linear algebraic equations to be solved for the ω_j ($j=1, 2, \ldots, N$) and H_2

$$\begin{aligned}&\omega_{j+1}-2\omega_j+\omega_{j+1}+\tfrac{1}{2}k^2(S_1+S_2)(H_2^2+H_1^2)\\&\quad+\varepsilon k^2(\omega_{j+1}-\omega_{j-1})[(3H_2^2+H_1^2)(\omega_1+\omega_2+\ldots+\tfrac{1}{2}\omega_j)-4(H_1^2+H_2^2)\delta_j]/8\Delta X\\&\quad-\varepsilon k^2\omega_j(\omega_j-2q_{1j})(H_1^2+H_2^2)/2\Delta X=0\end{aligned}\tag{12}$$

$$k(H_1+H_2)(\omega_1+\omega_2+\ldots+\tfrac{1}{2}\omega_N)-4=0\tag{13}$$

where $\delta_j=q_{11}+q_{12}+\ldots+\frac{1}{2}q_{1j}$. Equation (12) holds for $j=1, 2, \ldots, N$ and (11) gives $\omega_0=0$ and $\omega_{N+1}=\omega_{N-1}$.

Equations (12) and (13) have to be solved together by iteration using Newton's method. So that if we regard (12) and (13) as equations of the form $f_i(\omega_j; H_2)=0$ ($i=1, 2, \ldots, N+1$, and $j=1, 2, \ldots, N$) and $\omega_j^{(0)}$ and $H_2^{(0)}$ is an initial approximation to the solution, then a better approximation $\omega_j^{(0)}+\Delta\omega_j$ and $H_2^{(0)}+\Delta H_2$ is calculated by solving the linear equations

$$\left(\frac{\partial f_i}{\partial\omega_j}\right)_0\Delta\omega_j+\left(\frac{\partial f_i}{\partial H_2}\right)_0\Delta H_2=-f(\omega_j^{(0)}; H_2^{(0)}).$$

This process was repeated until the difference between the two solutions was sufficiently small (less than 10^{-5} for the present calculation). This method of iteration was found to converge quickly. Other methods of solving the non-linear equations were tried, such as linearising equation (12), or solving (12) first using an approximate value for H_2 and then trying to improve this by re-calculating it from (13) using the ω_j as calculated from (12). Both these methods were found to be unsuitable, as their convergence was extremely poor.

Errors introduced by taking differences in the X-direction were kept small by covering the step from X_1 to X_2 in first one then two steps and insisting that ΔX was small enough for the difference the two solutions thus obtained was small (less than $5\cdot10^{-4}$ in the present calculation). It was found by trial that errors introduced by using finite differences in the η-direction could be kept small enough by taking $N=50$, $k=0.02$. In this way an overall accuracy of at least three figures was achieved.

We can define a skin friction coefficient τ_ω by

$$\tau_\omega=\frac{\nu}{gh_0}\left(\frac{\partial u}{\partial y}\right)_0=\left(\frac{\partial^2\Psi}{\partial Y^2}\right)_0=\frac{1}{H}\left(\frac{\partial q}{\partial\eta}\right)_0.$$

$(\partial q/\partial\eta)_0$ was found using the formula

$$\left(\frac{\partial q_2}{\partial\eta}\right)_0=\frac{16q_2(k)-q_2(2k)+6k^2H_2^2S_2}{14k}$$

This was obtained by expanding q_2 in a Taylor series about $\eta=0$ and using the fact that $(\partial^2q_2/\partial\eta^2)_0=-H_2^2S_2$ and $(\partial^3q_2/\partial\eta^3)_0=0$.

In the case where the bed is a circular arc the numerical solution started at $X=0$ with the

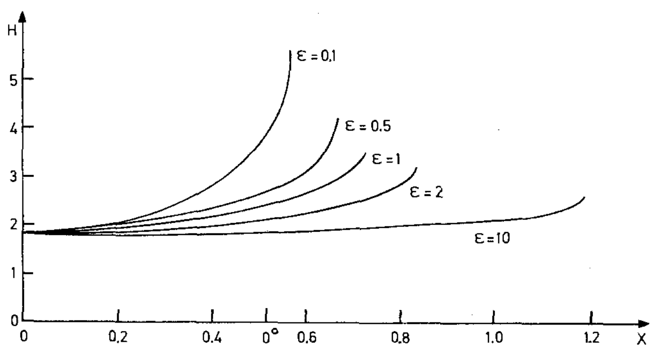

Figure 1. Values of film thickness H for various ε, bed profile a circular arc.

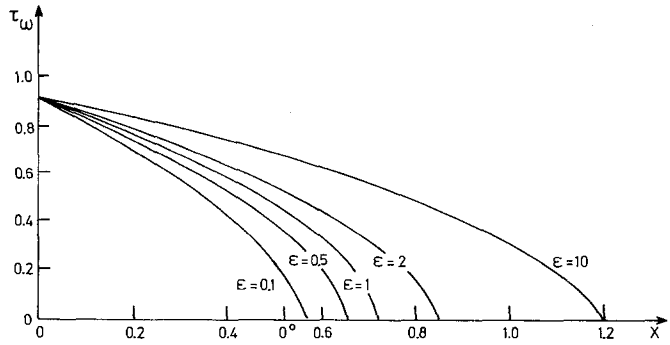

Figure 2. Values of skin friction τ_ω for various ε, bed profile a circular arc.

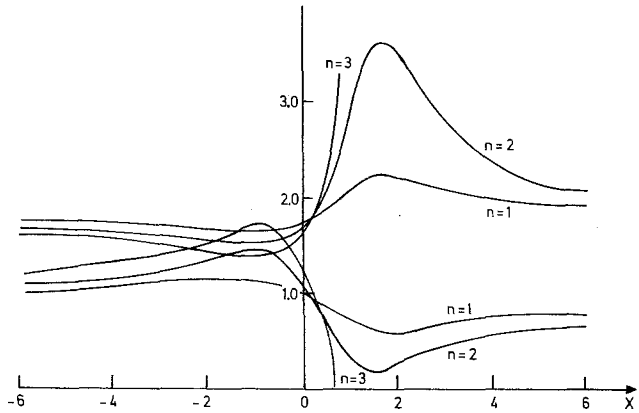

Figure 3. Values of film thickness H and skin friction τ_ω with bed profile $S_n = \frac{1}{2}(1 - nx/(1+x^2))$ for $n = 1$, 2 and 3. Higher curves are H, lower curves are τ_ω.

given initial conditions and proceeded downstream until the separation point (i.e. the point where τ_ω becomes zero) was reached. The calculation was performed for various ε with $\theta_0 = \pi/6$ and the values of H and τ_ω obtained are shown in figures 1 and 2 respectively. In figure 3 is given the values of H and τ_ω for $\varepsilon = 1$ when $S = S_n$ with $n = 1$, 2 and 3. In this case the integration started at a large negative value of X with the given initial conditions. As these hold only as $X \to -\infty$, this value of X had to be varied until one was found for which the numerical solution agreed with the asymptotic solution to within the required accuracy.

The numerical solution of the full equations enables us to verify the conjecture made by Smith [3]. He obtained approximate solution of equations (1) and (2) using a momentum integral technique. By using an integrated form of equation (2) and by making a suitable choice of velocity profile he reduced the problem to the solution of ordinary differential equations. He found that the solution of these equations broke down in certain cases, where, he conjectured, the flow was near a separation point. The exact solution confirms this. As $X \to X_S$ ($X = X_S$ is the separation point) the flow becomes singular in such a way that $\tau_\omega \to 0$ and $H \to H_S$ (where $H_S = H(X_S)$) while $d\tau_\omega/dX$ and dH/dX both$\to \infty$. So the breakdown in the solution found by Smith [3] is not dependent on the particular approximation used but appears in the exact solution. This singularity found in the solution of equations (9) and (10) is analogous to the singularity at separation found, for example, by Terrill [4], in the solution of the boundary-layer equations for the flow against an adverse pressure gradient and will be discussed in the next section.

Figures 1, 2 and 3 show that the flow separates only in regions of adverse slope, i.e. where $S(X) < 0$. Decreasing ε is to increase the importance of the viscous forces, but even with $\varepsilon = 0.1$ the flow separates at an angle of 2.3° past the horizontal, while for $\varepsilon = 10$ it is 38.8°. When $S = S_n(X)$, the flow separates when $n = 3$, but not when $n = 1$ and 2. In the latter case $S \geqq 0$

for all X, but in the former $S<0$ for $0.382<X<2.618$, the flow separating at $X=0.663$. This is in disagreement with Smith [3] who found that separation occurred in regions of favourable slope (i.e. where $S>0$). He found, for example that with $S=S_2$ the flow separated at $X=0.616$ whereas the exact solution shows that τ_ω has a minimum value at $X=1.5$. With $S>0$ gravity is accelerating the flow. This is analogous to the boundary-layer flow in a favourable pressure gradient. With $S<0$, however, the liquid film is flowing against gravity which will be retarding the flow (analogous to an adverse pressure gradient), and will reduce τ_ω to zero.

Also considered was the effect of varying the initial angle θ_0. The values of θ_S (the angle at which the flow separated) and H_S are given in table 1 for the case when the bed profile is a circular arc and $\varepsilon=1$. It is interesting to note that the angle past the vertical $(\theta_S-\theta_0)$ at which the flow separated is very nearly the same for all the θ_0 considered.

TABLE 1

θ_0	θ_S	H_S
30°	42.00	3.55
45°	57.53	3.50
60°	72.62	3.42
90°	102.73	3.33

4. Solution near the separation point

To discuss the solution near the separation point we first transform equation (6) by writing $\Psi=S_0^{\frac{1}{4}}\varepsilon^{-\frac{3}{4}}F(x_1, y_1)$ where $x_1=X_S-X$, $y_1=(\varepsilon S_0)^{\frac{1}{4}}Y$, and $S_0=-S(X_S)$ which will be positive since the flow will separate only in a region of adverse slope.

Equation (6) then becomes

$$\frac{\partial^3 F}{\partial y_1^3}+\frac{S(x_1)}{S_0}=\frac{\partial F}{\partial x_1}\frac{\partial^2 F}{\partial y_1^2}-\frac{\partial F}{\partial y_1}\frac{\partial^2 F}{\partial x_1 \partial y_1} \tag{14}$$

with boundary conditions

$$F=\frac{\partial F}{\partial y_1}=0 \quad \text{on} \quad y_1=0, \tag{15}$$

$$F=\varepsilon/b, \quad \frac{\partial^2 F}{\partial y_1^2}=0 \quad \text{on} \quad y_1=bH(x_1) \quad \text{where} \quad b=(\varepsilon S_0)^{\frac{1}{4}}. \tag{16}$$

The nature of the solution of equation (14) near a separation point has been discussed by Goldstein [5] and later by Stewartson [6] and Terrill [4]. Goldstein showed that, about the point $x_1=0$, $F(x_1, y_1)$ had the series expansion

$$F(x_1, y_1)=F_0(y_1)+a_1 x_1^{\frac{1}{4}}F_0'(y_1)+a_2 x_1^{\frac{3}{4}}F_0'(y_1) + a_3 x_1 \log x_1 F_0'(y_1)+x_1 F_4(y_1)+\dots \tag{17}$$

where dashes denote differentiation with respect to y_1 and where

$$F_4(y_1)=F_0'(y_1)\int_0^{y_1}\left(\frac{1-F_0'''+4\alpha_1^2(F_0'''F_0'-F_0''^2)}{F_0'^2}-\frac{2^{\frac{3}{2}}\pi\alpha_1^3}{(\frac{1}{4}!)^2}\cdot\frac{1}{t}\right)dt + \frac{2^{\frac{3}{2}}\pi\alpha_1^3}{(\frac{1}{4}!)^2}\log y_1\, F_0'(y_1)+a_4 F_0'(y_1).$$

The constants a_i $(i=1, 2, 3, 4)$ can all be expressed in terms of the one constant α_1 which, however, cannot be found from the series expansion. They are

$$a_1=2^{\frac{3}{2}}\alpha_1, \quad a_2=2^{\frac{7}{4}}\pi^{\frac{3}{2}}\alpha_1^2/5(\tfrac{1}{4}!)^3, \quad a_3=-\pi\alpha_1^3/2^{\frac{1}{2}}(\tfrac{1}{4}!)^2,$$

$$a_4 = \frac{\pi\alpha_1^3}{2^{\frac{1}{2}}(\frac{1}{4}!)^2}\left(\gamma - \frac{\pi}{2} - 5 + \frac{\pi^2}{100(\frac{1}{4}!)^4}(35-8\cdot 2^{\frac{1}{2}})\right).$$

γ is Euler's constant. $F_0(y_1)$, the separation profile, has an expansion for small y_1 in the form

$$F_0(y_1) = \frac{y_1^3}{6} - \frac{\alpha_1^2 y_1^5}{30} - \frac{\pi\alpha_1^3}{120\cdot 2^{\frac{1}{2}}(\frac{1}{4}!)^2}y_1^6 + \left[\left(\frac{4}{63} - \frac{\pi^2}{150(\frac{1}{4}!)^4}\right)\alpha_1^4 + \frac{S_1}{630}\right]\frac{y_1^7}{4} + \dots \tag{18}$$

where $S_1 = (S^{-1}dS/dx_1)_{x_1=0}$. The form of expansions (17) and (18) and the value of the constants a_i is determined from the condition that this solution must match with an inner solution. The details of which are given in Goldstein [5] and summarised in Stewartson [7]. (17) and (18) will contain terms which cannot be expressed in terms of the constant α_1 and (18) will also involve terms in log y_1, but these will be of higher order than the terms given in (17) and (18) (Brown and Stewartson [8]).

Using (18), the expansion (17) will satisfy the boundary conditions (15). In order to make (17) satisfy the boundary conditions (16) we must expand $H(x_1)$ in the form

$$H = H_S + H_1 x_1^{\frac{1}{2}} + H_2 x_1^{\frac{3}{2}} + H_3 x_1 \log x_1 + H_4 x_1 + \dots . \tag{19}$$

Putting this in (17) and using Taylor's theorem to expand the terms gives

$$F_0(bH_S) = \varepsilon/b, \quad F_0''(bH_S) = 0 \tag{20}$$

$$H_1 = -a_1/b, \quad H_2 - a_2/b, \quad H_3 = -a_3/b \tag{21}$$

and $H_4 = -F_4(bH_S)/bF_0'(bH_S)$. This last expression is obtained using $F_0''(bH_S)=0$. Thus we can set up a systematic scheme for finding the constants in (19). Since $F_0(y_1)$ involves the unknown constant α_1, the two equations given by (20) must be solved first to find α_1 and H_S. H_1, H_2 and H_3 can then be found knowing α_1, while H_S and F_0 are also needed to find H_4. The process could be carried on to higher terms if required.

To find α_1 and H_S exactly we need to know the function $F_0(y_1)$, but we know only its expansion for small y_1. The expansion (18) can, however, be used as an approximation for $F_0(y_1)$. Equations (20) then become

$$\begin{aligned} \frac{\varepsilon}{b} &= \frac{(bH_S)^3}{6}\left(1 - \frac{x^2}{5} - A_1 x^3 + A_2 x^4 + \frac{S_1}{420}(bH_S)^4\right), \\ 0 &= 1 - \tfrac{2}{3}x^2 - 5A_1 x^3 + 7A_2 x^4 + \frac{S_1(bH_S)^4}{60} \end{aligned} \tag{22}$$

where $A_1 = \pi/20\cdot 2^{\frac{1}{2}}\cdot(\frac{1}{4}!)^2$, $A_2 = 2\pi^2/2100(\frac{1}{4}!)^4$ and $x = \alpha_1 bH_S$. Equations (22) were solved

TABLE 2

X	H (exact)	H (approximate)	τ_ω (exact)	τ_ω (approximate)
0.7331	3.55	3.68	0.007	0.000
0.7325	3.51	3.62	0.015	0.014
0.7300	3.42	3.53	0.034	0.034
0.7100	3.13	3.21	0.107	0.108
0.6900	2.96	3.01	0.158	0.159
0.6500	2.72	2.70	0.240	0.242
0.6100	2.56	2.44	0.310	0.314
0.5700	2.43	2.22	0.373	0.379
0.5300	2.33	2.01	0.432	0.440
0.4900	2.24	1.82	0.486	0.497

TABLE 3

η	q (exact)	q (approximate)
0.1	0.015	0.014
0.2	0.056	0.055
0.3	0.118	0.119
0.4	0.195	0.199
0.5	0.282	0.286
0.6	0.368	0.368
0.7	0.446	0.436
0.8	0.508	0.482
0.9	0.548	0.503
1.0	0.562	0.506

together for α_1 and H_S, then these values used in (21) to find the H_i, (18) being used to calculate H_4. The values of H (given by (19)) and τ_ω (given by a similar series) near separation found in this way were compared with the exact values (given by the numerical solution) and there was good agreement in all the cases considered. For example, when $\theta_0 = \pi/6$ and $\varepsilon = 1$, the calculation gives $\alpha_1 = 0.539$ and $H_S = 3.655$. Table 2 gives a comparison between the approximate values of H and τ_ω calculated in this way and the exact values found from numerical solution.

There is difficulty in locating the exact separation point since the solution becomes singular there. The numerical solution stops just before separation with τ_ω still non-zero, though small. This final point is taken as the separation point for the purpose of the above calculation, and the results quoted are those at this point. The error in X_S in doing this will be small. In the above case X_S is taken as 0.7331, with the exact separation point being no more than 0.0003 further on. This will mean that the exact value of H_S will be higher than that given in table 2, perhaps increased by an estimated 0.04, though extrapolation here will be very unreliable. Also compared were the values of q as obtained from the numerical solution and (18). These values are given in table 3. The values from the numerical solution are again taken at the final step in the calculation. The values at the exact separation point will differ only slightly from those given in table 3.

REFERENCES

[1] E. J. Watson, The radial spread of a liquid jet over a horizontal plane, *J. Fluid Mech.*, 20 (1964) 481–99.
[2] R. C. Ackerberg, Boundary-layer flow on a vertical plate, *The Phys. of Fluids*, 11 (1968) 1278–91.
[3] S. H. Smith, Boundary-layer flow in a thin sheet of viscous liquid, *J. Eng. Maths.*, 5 (1971) 11–18.
[4] R. M. Terrill, Laminar boundary-layer flow near separation with and without suction, *Phil. Trans. Roy. Soc.*, A253 (1960) 55–100.
[5] S. Goldstein, On laminar boundary-layer flow near a position of separation, *Q.J.M.A.M.*, 1 (1948) 43–69.
[6] K. Stewartson, On Goldstein's theory of laminar separation, *Q.J.M.A.M.*, 11 (1958) 399–410.
[7] K. Stewartson, Is the singularity at separation removable? *J. Fluid Mech.*, 44 (1970) 347–64.
[8] S. N. Brown and K. Stewartson, Laminar separation, *Ann. Rev. Fluid Mech.*, 1 (1969) 45.